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$q = 0$ Oscillator Algebra as a Hopf Algebra

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In this paper we show that the $q = 0$ oscillator algebra is a Hopf algebra.

About 4 years ago Greenberg (1990) presented an example of infinite statistics for identical particles. He averaged the boson algebra

$$aa^+ - a^+a = 1 \tag{1}$$

and fermion algebra

$$aa^+ + a^+a = 1 \tag{2}$$

to get the new algebra

$$aa^+ = 1 \tag{3}$$

He showed that the statistical mechanics of particles satisfying the algebra (3) obeys the quantum Boltzmann statistics. After this work was done he found out that the algebras (1)–(3) are special examples of a q-deformed boson algebra (Biedenharn, 1989; Macfarlane, 1989; Arik and Coon, 1976),[2]

$$\begin{aligned} aa^+ - qa^+a &= 1 \\ [N, a^+] &= a^+ \\ [N, a] &= -a \end{aligned} \tag{4}$$

where N is a number operator.

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[2]When $q = 1, -1, 0$, the algebra (4) reduces to the algebras (1), (2), and (3), respectively.

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Greenberg assumed the existence of a unique vacuum state annihilated by the annihilation operator a:

$$a|0\rangle = 0 \tag{5}$$

He constructed the number operator N for the algebra (3) satisfying the second and third relations of (4), whose form is given by

$$\begin{aligned} N &= a^+a + a^+a^+aa + a^+a^+a^+aaa + \cdots \\ &= \sum_{m=1}^{\infty} (a^+)^m a^m \end{aligned} \tag{6}$$

Here we check that the number operator (6) satisfies the second and third relations of (4):

$$\begin{aligned} [N, a^+] &= \sum_{m=1}^{\infty} (a^+)^m a^m a^+ - \sum_{m=1}^{\infty} (a^+)^{m+1} a^m \\ &= \sum_{m=1}^{\infty} (a^+)^m a^{m-1} aa^+ - a^+ \sum_{m=1}^{\infty} (a^+)^m a^m \\ &= a^+ \left(\sum_{m=1}^{\infty} (a^+)^{m-1} a^{m-1} - \sum_{m=1}^{\infty} (a^+)^m a^m \right) \\ &= a^+ \left(1 + \sum_{m=2}^{\infty} (a^+)^{m-1} a^{m-1} - \sum_{m=1}^{\infty} (a^+)^m a^m \right) \\ &= a^+ \left(1 + \sum_{m=1}^{\infty} (a^+)^m a^m - \sum_{m=1}^{\infty} (a^+)^m a^m \right) \\ &= a^+ \end{aligned} \tag{7}$$

which implies that the number operator given in (6) fulfills the properties for the number operator. This kind of number operator cannot be obtained by setting $q = 0$ in the number operator given in the q-boson algebra. The number operator for the q-boson algebra (4) is already known from

$$[N] = \frac{q^N - 1}{q - q^{-1}} = a^+a \tag{8}$$

Substituting $q = 0$ in equation (7) leads to no relation between the number operator N and mode operators a and a^+, because a q-number goes to 1 whenever q goes to 0.

Now we will show that this algebra ($q = 0$ oscillator algebra) is a Hopf algebra. The three operations of the Hopf algebra for the $q = 0$ oscillator algebra are given by

$$\Delta(N) = N \otimes I + I \otimes N$$
$$\Delta(I) = I \otimes I$$
$$\Delta(a^+) = I \otimes a^+$$
$$\Delta(a) = I \otimes a$$
$$\epsilon(I) = 1, \qquad \epsilon(a) = a, \qquad \epsilon(a^+) = a^+, \qquad \epsilon(N) = 0$$
$$S(I) = 1, \qquad S(a) = a, \qquad S(a^+) = a^+, \qquad S(N) = -N$$

where Δ, S, and ϵ are called the coproduct, antipode, and counit, respectively. The Hopf algebra A should satisfy the following three axioms for all elements (generators) $x \in A$:

$$(\mathrm{id} \otimes \Delta)\Delta(x) = (\Delta \otimes \mathrm{id})\Delta(x)$$
$$m(\mathrm{id} \otimes S)\Delta(x) = m(S \otimes \mathrm{id})\Delta(x) = \epsilon(x) \cdot 1$$
$$(\epsilon \otimes \mathrm{id})\Delta(x) = (\mathrm{id} \otimes \epsilon)\Delta(x) = x, \qquad x = I, a, a^+, N$$

It is easy to check that this set of definitions fulfills the three axioms. First let us prove that the coproduct axiom holds for a:

$$(\mathrm{id} \otimes \Delta)\Delta(a) = (\Delta \otimes \mathrm{id})\Delta(a)$$

Proof. LHS $= (\mathrm{id} \otimes \Delta)\Delta(a) = (\mathrm{id} \otimes \Delta)(I \otimes a) = I \otimes \Delta(a) = I \otimes I \otimes a$; RHS $= (\Delta \otimes \mathrm{id})\Delta(a) = (\Delta \otimes I)(I \otimes a) = \Delta(I) \otimes a = I \otimes I \otimes a$.

Second let us check the case for N:

$$(\mathrm{id} \otimes \Delta)\Delta(N) = (\Delta \otimes \mathrm{id})\Delta(N)$$

Proof. LHS $= (\mathrm{id} \otimes \Delta)(N \otimes I + I \otimes N) = N \otimes I \otimes I + I \otimes N \otimes I + I \otimes I \otimes N$; RHS $= (\Delta \otimes \mathrm{id})(N \otimes I + I \otimes N) = N \otimes I \otimes I + I \otimes N \otimes I + I \otimes I \otimes N$.

The counit axiom is verified as follows:

$$(\epsilon \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \epsilon)\Delta(a) = a$$

Proof. LHS $= (\epsilon \otimes \mathrm{id})\Delta(a) = (\epsilon \otimes \mathrm{id})(I \otimes a) = \epsilon(I) \otimes a = a$; RHS $= (\Delta \otimes \epsilon)\Delta(a) = (\mathrm{id} \otimes \epsilon)(I \otimes a) = I \otimes \epsilon(a) = a$.

We will prove that

$$m(\mathrm{id})\Delta(a) = m(S \otimes \mathrm{id})\Delta(a) = \epsilon(a)$$

Proof. LHS $= m(\mathrm{id} \otimes S)\Delta(a) = I \cdot S(a) = a = \epsilon(a)$; RHS $= m(S \otimes \mathrm{id})\Delta(a) = S(I)a = a = \epsilon(a)$.

We now verify that

$$\Delta(aa^{+}) = \Delta(a)\Delta(a^{+})$$

Proof. LHS $= \Delta(aa^{+}) = \Delta(I) = I \otimes I$; RHS $= (I \otimes a)(I \otimes a^{+}) = I \otimes aa^{+} = I \otimes I$.

It is easy to check that the following formula holds:

$$\Delta([N, a]) = [\Delta(N), \Delta(a)]$$

Proof. $[\Delta(N), \Delta(a)] = [N \otimes I + I \otimes N, I \otimes a] = N \otimes a - N \otimes a + I \otimes Na - I \otimes aN = I \otimes [N, a] = -I \otimes a = -\Delta(a) = \Delta(-a) = \Delta([N, a])$.

From the above proofs, we can conclude that $q = 0$ oscillator algebra is really a Hopf algebra.

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